Exercise 16C: Solutions

1 a

$$\begin{bmatrix} \cos 270 & -\sin 270 \\ \sin 270 & \cos 270 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

b

$$\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

c

$$\begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

d

$$\begin{bmatrix} \cos(-135^\circ) & -\sin(-135^\circ) \\ \sin(-135^\circ) & \cos(-135^\circ) \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2 a The rotation matrix is

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

so that $(2, 3) \to (-3, 2)$.

b The rotation matrix is $\begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ Therefore

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

so that $(2, 3) \to \left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

3 a

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ \sin 90^\circ & -\cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

b

$$\begin{bmatrix} \cos 120^\circ & \sin 120^\circ \\ \sin 120^\circ & -\cos 120^\circ \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

c

$$\begin{bmatrix} \cos(-60^\circ) & \sin(-60^\circ) \\ \sin(-60^\circ) & -\cos(-60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

d

$$\begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ \sin 30^\circ & -\cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

4 a Since

$$\tan\theta = 3 = \frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 3 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{10}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = \frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1 \\ &= \frac{2}{10} - 1 \\ &= -\frac{4}{5}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}} \\ &= \frac{3}{5}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

b Since

$$\tan\theta = 5 = \frac{5}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 5 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{26}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{26}} \text{ and } \sin\theta = \frac{5}{\sqrt{26}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{26}}\right)^2 - 1 \\ &= \frac{2}{26} - 1 \\ &= -\frac{12}{13}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{1}{\sqrt{26}}\frac{5}{\sqrt{26}} \\ &= \frac{5}{13}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}.$$

c Since $\tan\theta = \frac{2}{3}$,

we draw a right angled triangle with opposite and adjacent lengths 2 and 3 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{13}$. Therefore,

$$\cos\theta = \frac{3}{\sqrt{13}} \text{ and } \sin\theta = \frac{2}{\sqrt{13}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{3}{\sqrt{13}}\right)^2 - 1 \\ &= \frac{18}{13} - 1 \\ &= \frac{5}{13}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{2}{\sqrt{13}}\frac{3}{\sqrt{13}} \\ &= \frac{12}{13}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}.$$

d Since

$$\tan\theta = -3 = -\frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 3 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{10}$. Therefore, since $-90^\circ < \theta < 0^\circ$,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = -\frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1 \\ &= \frac{2}{10} - 1 \\ &= -\frac{4}{5}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= -2\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}} \\ &= -\frac{3}{5}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

5 **a** Since

$$\tan\theta = m = \frac{m}{1},$$

we draw a right angled triangle with opposite and adjacent lengths m and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{m^2+1}$. Therefore,

$$\begin{aligned} \cos\theta &= \frac{1}{\sqrt{m^2+1}} \\ \sin\theta &= \frac{m}{\sqrt{m^2+1}}. \end{aligned}$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{m^2+1}}\right)^2 - 1 \\ &= \frac{2}{m^2+1} - 1 \\ &= \frac{1-m^2}{m^2+1}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned}$$

$$=2\frac{m}{\sqrt{m^2+1}}\frac{1}{\sqrt{m^2+1}}$$
$$=\frac{2m}{m^2+1}.$$

Therefore the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}=\begin{bmatrix}\frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1}\end{bmatrix}.$$

b The gradient of the line is $m=6$. Substituting this into the matrix found above, the reflection matrix is

$$\begin{bmatrix}\frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & -\frac{m^2-1}{m^2+1}\end{bmatrix}=\begin{bmatrix}\frac{1-6^2}{6^2+1} & \frac{2\times 6}{6^2+1} \\ \frac{2\times 6}{6^2+1} & -\frac{6^2-1}{6^2+1}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{-35}{37} & \frac{12}{37} \\ \frac{12}{37} & -\frac{35}{37}\end{bmatrix}$$
$$=\frac{1}{37}\begin{bmatrix}-35 & 12 \\ 12 & 35\end{bmatrix}$$

Therefore the image of $(1,1)$ can be found by evaluating,

$$\frac{1}{37}\begin{bmatrix}-35 & 12 \\ 12 & 35\end{bmatrix}\begin{bmatrix}1 \\ 1\end{bmatrix}=\frac{1}{37}\begin{bmatrix}-23 \\ 47\end{bmatrix}$$

so that

$$(1,1)\to\left(\frac{-23}{37},\frac{47}{37}\right).$$

6 a

$$\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

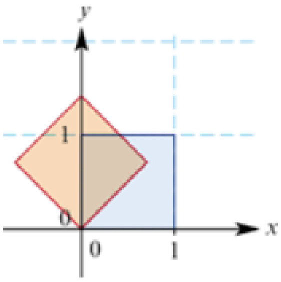
b To find the image of the unit square we evaluate

$$\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1\end{bmatrix}$$
$$=\begin{bmatrix}0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2}\end{bmatrix}.$$

The columns then give the required points:

$$(0,0),\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),(0,\sqrt{2}).$$

The square is shown in blue, and its image in red.



c To find the overlapping region, we subtract the area of the small upper isosceles triangle from the right half of the red square. The base and height of the small isosceles triangle is $\sqrt{2}-1$ so that the overlapping area is

$$\begin{aligned} A &= \frac{1}{2} - \frac{1}{2}(\sqrt{2}-1)^2 \\ &= \frac{1}{2} - \frac{1}{2}(2-2\sqrt{2}+1) \\ &= \frac{1}{2} - \frac{1}{2}(3-2\sqrt{2}) \\ &= \frac{1}{2} - \frac{3}{2} + \sqrt{2} \\ &= \sqrt{2}-1. \end{aligned}$$

7 a There is no real need to use the rotation matrix for this question. Let O be the origin. We know that length $OA = 1$. Therefore, lengths $OB = 1$ and $OC = 1$. Therefore,

$$B = (\cos 120^\circ, \sin 120^\circ) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$C = (\cos 240^\circ, \sin 240^\circ) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

b Triangle ABC is clearly equilateral.

c Its lines of symmetry will be

$y = x\tan 60^\circ = \sqrt{3}x$

$y = 0$

$y = x\tan 300^\circ = -\sqrt{3}x$